REAL ANALYSIS
TOPIC 32 - THE DARBOUX INTEGRAL

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Abstract. We describe the modern definition of the Riemann integral, which is a little easier to use in formal proofs. We will call this the Darboux integral, to distinguish it from the previous definition. The definition of the Riemann integral previously given is equivalent to the definition of the Darboux integral given here, as to which functions are integrable, and the value of the definite integral.

1. Darboux Integral

We develop and alternate definition of the integral,

1.1. Partitions.

Definition 1. Let $a, b \in \mathbb{R}$ with $a < b$. A *partition* of the closed interval $[a, b]$ is a finite set

$$P = \{x_0, x_1, x_2, \dots, x_n\}$$

with the property that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

We write a partition P as a set, but it is in fact an ordered set, and by convention, the order is dictated by the indices of the points in the set. We view P as indicating a way of breaking the interval $[a, b]$ into n closed subintervals, $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$.

Definition 2. Let $a, b \in \mathbb{R}$ with $a < b$. Let P be a partition of $[a, b]$. A *refinement* of P is a partition Q of $[a, b]$ such that $P \subset Q$.

Proposition 1. *Any two partitions of* $[a, b]$ *have a common refinement.*

Proof. Let P_1 and P_2 be partitions of $[a, b]$. Then $Q = P_1 \cup P_2$ is a refinement of P_1 and of P_2. □

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1.2. **Darboux Sums.**

Definition 3. Let f be a bounded function defined on a closed interval $[a, b]$. Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be a partition of $[a, b]$. Set

$m_k = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}, M_k = \sup\{f(x) \mid x \in [x_{i-1}, x_i],$ and $\Delta x_i = x_i - x_{i-1}\}$, for $i = 1, \ldots, n$.

The *lower Darboux sum* of f over P is

$$\underline{S}(f, P) = \sum_{i=1}^{n} m_i \, \Delta x_i.$$

The *upper Darboux sum* of f for P is

$$\overline{S}(f, P) = \sum_{i=1}^{n} M_i \, \Delta x_i.$$

It is clear from the definition that $\underline{S}(f, P) \leq \overline{S}(f, P)$.

Proposition 2. *Let f be a bounded function defined on a closed interval $[a, b]$. Let P be a partition of $[a, b]$, and let Q be a refinement of P. Then*

$$\underline{S}(f, P) \leq \underline{S}(f, Q) \leq \overline{S}(f, Q) \leq \overline{S}(f, P).$$

Proof. We discuss the last two inequalities, the first one being similar to the last.

Consider the middle inequality. It states that $\sum_{i=1}^{n} m_i \, \Delta x_i \leq \sum_{i=1}^{n} M_i \, \Delta x_i$. But this is clear, since $m_i = \inf A \leq \sup A = M_i$ for $A = f([x_{i-1}, x_i])$.

Consider the last inequality. If $P = Q$, we have equality here. Otherwise, Q contains at least one more point than P; let us suppose that Q contains exactly one more point than P. This point is in one of the subintervals determined by P, say $y \in Q$ and $x_{i-1} < y < x_i$. Then

$$\begin{aligned}
\overline{S}(f, P) - \overline{S}(f, Q) &= (\sup f([x_{i-1}, x_i]))(x_i - x_{i-1}) \\
&\quad - (\sup f([x_{i-1}, y]))(y - x_{i-1}) + (\sup f([y, x_i]))(x_i - y) \\
&= (\sup f([x_{i-1}, x_i]) - \sup(f([x_{i-1}, y])))(y - x_{i-1}) \\
&\quad + (\sup f([x_{i-1}, x_i]) - (\sup f([y, x_i])))(x_i - y) \\
&\geq 0,
\end{aligned}$$

since $B \subset A$ implies $\sup A \geq \sup B$. Since the inequality holds if we add one point to P, it will continue to hold as we add more points. $\square$

1.3. **Darboux Integral.**

Definition 4. Let f be a bounded function defined on a closed interval $[a,b]$.

The *lower Darboux integral* of f on $[a,b]$ is

$$\underline{\int_a^b} f = \sup\{\underline{S}(f,P) \mid P \text{ is a partition of } [a,b]\}.$$

The *upper Darboux integral* of f on $[a,b]$ is

$$\overline{\int_a^b} f = \inf\{\overline{S}(f,P) \mid P \text{ is a partition of } [a,b]\}.$$

Many books call these the lower and upper Riemann integral.

Proposition 3. *Let f be a bounded function defined on a closed interval $[a,b]$. Let P be a partition of $[a,b]$. Let $m = \inf\{f(x) \mid x \in [a,b]$ and $M = \sup\{f(x) \mid x \in [a,b]\}$. Then*

$$m(b-a) \leq \underline{S}(f,P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq \overline{S}(f,P) \leq M(b-a).$$

Proof. We discuss the last three inequalities, as the first two are analogous to the last two.

The last inequality is obtained by setting $Q = \{a,b\}$, so that P is a refinement of Q. Then

$$\overline{S}(f,P) \leq \overline{S}(f,Q) = M(b-a).$$

That $\overline{\int_a^b} f \leq \overline{S}(f,P)$ follows from the fact that $\overline{\int_a^b} f$ is the supremum of a set which contains $\overline{S}(f,P)$.

That $\underline{\int_a^b} f \leq \overline{\int_a^b} f$ follows from the fact that if $a \leq b$ for every $a \in A$ and $b \in B$, then $\sup A \leq \inf B$. ☐

Definition 5. Let f be a bounded function defined on a closed interval $[a,b]$. We say that f is *Darboux integrable* on $[a,b]$ if

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

In this case, the common value is called the *Riemann integral* of f on $[a,b]$, and is denoted

$$\int_a^b f.$$

It can be shown that a function is Riemann integrable if and only if it is Darboux integrable.

Example 1. Define a function $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational ;} \\ 1 & \text{if } x \text{ is rational .} \end{cases}$$

Every subinterval of every partition contains an open interval which contains both rational and irrational numbers. Thus $m_k = 0$ and $M_k = 1$ for all subintervals, whence $\underline{\int_0^1} f = 0$ and $\overline{\int_0^1} f = 1$. Thus f is not Darboux integrable.

2. Exercises

Problem 1. Show that f is Darboux integrable on $[a, b]$ if and only if, for every $\epsilon > 0$ there exists a partition P of $[a, b]$ such that

$$|\overline{S}(f, P) - \underline{S}(f, P)| < \epsilon.$$

Problem 2. If $r \in \mathbb{Q}$, there exists $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $r = \frac{p}{q}$. Define $q : \mathbb{Q} \to \mathbb{R}$ by

$$q(r) = \min\{q \in \mathbb{N} \mid r = \frac{p}{q} \text{ for some } p \in \mathbb{Z}\}.$$

Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{q(x)} & \text{if } x \text{ is rational} \end{cases}$$

Is f Darboux integrable on $[0, 1]$? If so, what is the value of the definite integral?

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